Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 320 (2008) 48–56
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Resolutions over symmetric algebras with radical cube zero

David J. Benson

Department of Mathematics, University of Aberdeen, Aberdeen, AB24 3UE, UK

Received 28 August 2007

Available online 24 March 2008

Communicated by Luchezar L. Avramov

Abstract

Let Λ be a finite dimensional indecomposable weakly symmetric algebra over an algebraically closed field k, satisfying $J^3(\Lambda)=0$. Let $S_1,\dots,S_r$ be representatives of the isomorphism classes of simple Λ-modules, and let E be the $r\times r$ matrix whose (i,j) entry is $\dim_k \operatorname{Ext}^1_\Lambda(S_i,S_j)$. If there exists an eigenvalue λ of E satisfying $|\lambda|>2$ then the minimal resolution of each non-projective finitely generated Λ-module has exponential growth, with radius of convergence $\frac{1}{2}(\lambda-\sqrt{\lambda^2-4})$. On the other hand, if all eigenvalues λ of E satisfy $|\lambda|\leqslant 2$ then the dimensions of the modules in the minimal projective resolution of each finitely generated Λ-module are either bounded or grow linearly. In this case, we classify the possibilities for the matrix E. The proof is an application of the Perron–Frobenius theorem.

Keywords: Representations of algebras; Projective resolutions

1. Introduction

In the course of investigating the homology of the loop space on the p-completion of the classifying space of a finite group [3], the question of the rate of growth of resolutions over certain finite dimensional symmetric algebras became important. This paper is a small contribution to that subject.

The purpose of this paper is to describe minimal projective resolutions of modules over a finite dimensional symmetric algebra Λ satisfying $J^3(\Lambda)=0$. In fact, the proof only uses the

E-mail address: bensondj@maths.abdn.ac.uk.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.02.033

fact that Λ is weakly symmetric, so we formulate the theorem in that context. Recall that a finite dimensional self injective algebra Λ is said to be *weakly symmetric* if $P/\operatorname{Rad}(P) \cong \operatorname{Soc}(P)$ for each projective Λ-module P.

Let k be an algebraically closed field, and let Λ be a finite dimensional indecomposable weakly symmetric k-algebra satisfying $J^3(\Lambda) = 0$. Let $S_1, \dots, S_r$ a set of representatives for the isomorphism classes of simple Λ-modules. Write E for the Ext matrix

$$E_{i,j} = \dim_k \operatorname{Ext}^1_\Lambda(S_i, S_j).$$

Our main theorem is as follows.

Theorem 1.1. *The matrix E is symmetric, so its eigenvalues are real. The eigenvalue λ with largest absolute value is positive, and is a simple root of the characteristic equation of E.*

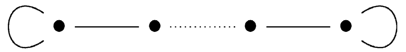
(i) *If $\lambda > 2$ then the minimal projective resolution of each finitely generated non-projective Λ-module has exponential growth. The Poincaré series*

$$\sum_{n=0}^{\infty} t^n \dim_k \Omega^n M$$

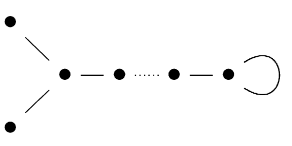
is the power series expansion of a rational function of t whose radius of convergence R is independent of the module. The radius R is determined by λ, and is given by

$$R = \frac{1}{2}\left(\lambda - \sqrt{\lambda^2 - 4}\right).$$

(ii) *If $\lambda = 2$ then the dimensions of the modules in the minimal projective resolution of each finitely generated Λ-module are either bounded or grow linearly. The only modules of linear growth are the modules $\Omega^m(S_i)$ with $m \in \mathbb{Z}$. The matrix E is the adjacency matrix of a Euclidean diagram $\tilde{A}_n$ $(n \geqslant 1)$, $\tilde{D}_n$ $(n \geqslant 4)$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, the graph*

$\tilde{Z}_n =$  ($n+1$ *nodes*, $n \geqslant 0$)

(*the adjacency matrix of $\tilde{Z}_0$ is the 1×1 matrix* (2)) *or the graph*

$\widetilde{DZ}_n =$  ($n+1$ *nodes*, $n \geqslant 2$).

(iii) *If $\lambda < 2$ then the dimensions of the modules in the minimal projective resolution of each finitely generated non-projective Λ-module are bounded. The matrix E is the adjacency matrix of a Dynkin diagram A_n $(n \geqslant 1)$, D_n $(n \geqslant 4)$, E_6, E_7, E_8, or the graph*

$Z_n =$  (n *nodes*, $n \geqslant 1$)

(*the adjacency matrix of Z_1 is the 1×1 matrix* (1)).

We remark that if B is a block of the modular group algebra of a finite group, and $J^3(B) = 0$, then Okuyama [9] has determined all the possibilities. Only cases (ii) and (iii) can occur, because in this situation exponential growth is not possible. Case (ii) only occurs for blocks with Klein four group as defect group in characteristic two, in which case the block is Morita equivalent to the group algebra of either the defect group or the alternating group of degree four. Case (iii) only occurs for blocks with cyclic defect group with inertial index $p-1$ or $(p-1)/2$, for which the Brauer tree is a straight line and the exceptional vertex, if there is one, is at an end.

We generalise Okuyama's theorem to finite dimensional cocommutative Hopf algebras (i.e., finite group schemes) as follows.

Corollary 1.2. *If a block Λ of a finite dimensional cocommutative Hopf algebra satisfies $J^3(\Lambda) = 0$ then all eigenvalues λ of the Ext matrix E are real numbers satisfying $|\lambda| \leqslant 2$. The conclusion in case* (ii) *of Theorem* 1.1 *holds if $\lambda = 2$ is an eigenvalue of E, and the conclusion in case* (iii) *of the theorem holds otherwise.*

We also remark that it is possible to generalise the main theorem in various directions. If we allow fields k that are not algebraically closed, then the other Dynkin and Euclidean diagrams come into play. It is also possible to generalise the main theorem to finite dimensional self injective algebras Λ satisfying $J^3(\Lambda) = 0$. In this case, E is no longer symmetric, but its transpose is equal to its product with a permutation matrix corresponding to the Nakayama functor, and E commutes with this permutation matrix.

Finally, we remark that in some sense, our main theorem is not really new. What is new is the short and direct proof. For a proof using results already available in the literature, one might argue as follows. The map $\Lambda \to \Lambda/J^2$ induces a functor from Λ/J^2-modules to Λ-modules and a bijection from all indecomposable Λ/J^2-modules to non-projective indecomposable Λ-modules. Next, we form a hereditary algebra with twice as many simple modules: $\Gamma = \left(\begin{smallmatrix} \Lambda/J & J/J^2 \\ 0 & \Lambda/J \end{smallmatrix}\right)$. It is shown in Corollary 2.7 of Reiten [11] that the non-projective indecomposable Λ/J^2-modules are in bijection with the non-projective indecomposable Γ-modules. Both of these correspondences preserve the Auslander–Reiten translation. The trichotomy of Theorem 1.1 corresponds exactly to the trichotomy between finite, tame and wild representation type for Γ, see for example Dlab and Ringel [4], and hence also for Λ.

2. Minimal resolutions over Λ

Let Λ be a finite dimensional indecomposable weakly symmetric algebra satisfying $J^3(\Lambda) = 0$. If $J^2(\Lambda) = 0$ then $E = (1)$, and every indecomposable module is either projective, or simple and periodic. So we are trivially in case (iii) of the main theorem. Therefore, from now on we assume that $J^2(\Lambda) \neq 0$.

Let $S_1, \dots, S_r$ be representatives for the isomorphism classes of simple Λ-modules, and let P_i be the projective cover of S_i. Then P_i is a module of Loewy length three with socle $\mathrm{Soc}(P_i) \cong S_i$.

If M is a finite dimensional Λ-module, we write $\underline{\dim}\, M$ for the *dimension vector* of M, namely a column vector of length r whose ith component is the multiplicity of S_i as a composition factor of M. If M is indecomposable then there are three possibilities. If M is simple then $\mathrm{Rad}(M) = 0$. If M is projective then $\mathrm{Rad}^2(M) \cong M/\mathrm{Rad}(M)$. If M is neither simple nor projective then $\mathrm{Soc}(M) = \mathrm{Rad}(M)$ and $\mathrm{Rad}^2(M) = 0$. In this case, we associate to M a column vector $d(M)$ of length $2r$, in which the first r components give the dimension vector $\alpha(M) = \underline{\dim}\, M/\mathrm{Rad}(M)$, and whose last r components give the dimension vector $\beta(M) = \underline{\dim}\,\mathrm{Rad}(M)$:

$$d(M)=\begin{pmatrix}\alpha(M)\\ \beta(M)\end{pmatrix}=\begin{pmatrix}\underline{\dim} M/\operatorname{Rad}(M)\\ \underline{\dim}\operatorname{Rad}(M)\end{pmatrix}.$$

The projective cover of M has radical layers with dimension vectors $\alpha(M)$, $E\alpha(M)$, $\alpha(M)$. So provided ΩM is not a simple module, we have $\alpha(\Omega M)=E\alpha(M)-\beta(M)$ and $\beta(\Omega M)=\alpha(M)$. So

$$d(\Omega M)=\begin{pmatrix}E\alpha(M)-\beta(M)\\ \alpha(M)\end{pmatrix}=\begin{pmatrix}E & -I\\ I & 0\end{pmatrix}\begin{pmatrix}\alpha(M)\\ \beta(M)\end{pmatrix}.$$

Repeating, we have

$$d\big(\Omega^n M\big)=\begin{pmatrix}E & -I\\ I & 0\end{pmatrix}^n d(M) \tag{2.1}$$

provided that none of the $\Omega^n M$ are simple modules.

On the other hand,

$$\begin{pmatrix}E & -I\\ I & 0\end{pmatrix} d\big(\Omega^{-1}S_i\big)=\begin{pmatrix}0\\ \underline{\dim}(S_i)\end{pmatrix},$$

while

$$d(\Omega S_i)=\begin{pmatrix}E & -I\\ I & 0\end{pmatrix}\begin{pmatrix}\underline{\dim}(S_i)\\ 0\end{pmatrix}.$$

Note also that

$$\begin{pmatrix}E & -I\\ I & 0\end{pmatrix}\begin{pmatrix}0\\ \underline{\dim}(S_i)\end{pmatrix}=\begin{pmatrix}-\underline{\dim}(S_i)\\ 0\end{pmatrix},$$

and that the inverse of the matrix $\left(\begin{smallmatrix}E & -I\\ I & 0\end{smallmatrix}\right)$ is the matrix $\left(\begin{smallmatrix}0 & I\\ -I & E\end{smallmatrix}\right)$.

3. Eigenvalues

The Perron–Frobenius theorem is a very useful theorem in the context of Ext matrices of indecomposable algebras. We state it in its general form, even though we do not use the full force of it in this paper.

Definition 3.1. We say that an $r\times r$ matrix A of non-negative real numbers is *connected* if there is no partition of $\{1,\dots,r\}$ into non-empty subsets I and J in such a way that $A_{i,j}=A_{j,i}=0$ for all $i\in I$, $j\in J$. This is equivalent to the condition that every entry in the matrix $I+A+A^2+\cdots+A^{r-1}$ is strictly positive. The Ext matrix E of an indecomposable finite dimensional algebra is connected; see Proposition II.5.2 of [1].

Theorem 3.2 *(Perron–Frobenius). Let A be a connected $r\times r$ matrix of non-negative real numbers. Enumerate the eigenvalues $\lambda_1,\dots,\lambda_r$ so that*

$$|\lambda_1|=\cdots=|\lambda_h|>|\lambda_{h+1}|\geqslant\cdots\geqslant|\lambda_r|\quad(h\geqslant 1).$$

Then

(i) *The positive real number* $\lambda = |\lambda_1|$ *is a simple root of the characteristic equation of* A. *In other words, the corresponding algebraic eigenspace is one dimensional.*
(ii) *There exists an eigenvector* v *of* A *with all entries positive, with eigenvalue* λ.
(iii) *After reordering if necessary, the numbers* $\lambda_1, \dots, \lambda_h$ *are equal to* $\lambda, \omega\lambda, \dots, \omega^{h-1}\lambda$ *where* $\omega = e^{2\pi i/h}$.
(iv) *The product of any eigenvalue of* A *with* ω *is another eigenvalue of* A.
(v) *If* $h > 1$ *then after conjugating by a permutation matrix if necessary,* A *has the form*

$$\begin{pmatrix} 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A_{h-1} \\ A_h & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where each A_i *is an* $r/h \times r/h$ *matrix.*

Proof. Perron [10] proved parts (i) and (ii) for matrices with all entries strictly positive. Frobenius [6] proved the full form of the theorem. $\square$

Remark 3.3. If the matrix A in the theorem happens also to be symmetric, then we are forced to have $h = 1$ or $h = 2$. In other words, there is a positive real eigenvalue λ_1 of A such that for all $i > 1$ we have $\lambda_1 > |\lambda_i|$, with the possible exception of a single eigenvalue $\lambda_2 = -\lambda_1$ if $h = 2$. We call this eigenvalue λ_1 the *dominant eigenvalue*.

We now return to the situation of the last section. We apply the Perron–Frobenius theorem to the Ext matrix E to obtain a dominant eigenvalue λ. If v is an eigenvector of E with eigenvalue λ, and μ satisfies $\mu^2 - \lambda\mu + 1 = 0$, then $\binom{\mu v}{v}$ is an eigenvector of $\left(\begin{smallmatrix} E & -I \\ I & 0 \end{smallmatrix}\right)$ with eigenvalue μ. So we investigate the solutions μ of this quadratic equation.

Lemma 3.4. *Let* λ *be a positive real number. If* μ *is one solution of* $\mu^2 - \lambda\mu + 1 = 0$ *then the other solution is* μ^{-1}, *and we have* $\lambda = \mu + \mu^{-1}$.

(i) *If* $\lambda > 2$ *then* μ *and* μ^{-1} *are distinct real numbers.*
(ii) *If* $\lambda = 2$ *then* $\mu = 1$ (*with multiplicity two*).
(iii) *If* $\lambda < 2$ *then* μ *and* μ^{-1} *are non-real complex numbers with absolute value one.*

Proof. This is straightforward. $\square$

Since E is a symmetric matrix, the algebraic and geometric eigenspaces of E coincide. If v_i is an eigenvector of E with eigenvalue λ_i, and μ_i satisfies the quadratic equation $\mu_i^2 - \lambda_i\mu_i + 1 = 0$, then $\binom{\mu_i v_i}{v_i}$ is an eigenvector of $\left(\begin{smallmatrix} E & -I \\ I & 0 \end{smallmatrix}\right)$ with eigenvalue μ_i. Provided $\lambda_i \neq 2$, the quadratic equation gives two different values of μ_i. If $\lambda_i = 2$ then $\mu_i = 1$ is the only solution, but the eigenspace contains not only the geometric eigenvector $\binom{v_i}{v_i}$ but also the algebraic eigenvector $\binom{v_i}{0}$. In either case, we get two dimensions of algebraic eigenvectors of $\left(\begin{smallmatrix} E & -I \\ I & 0 \end{smallmatrix}\right)$ for each dimension of eigenvectors of E, but it is only for $\lambda_i = 2$, $\mu_i = 1$ that the algebraic and geometric eigenspaces of $\left(\begin{smallmatrix} E & -I \\ I & 0 \end{smallmatrix}\right)$ differ.

If $|\lambda_i| \leqslant 2$ then both values of μ_i satisfy $|\mu_i| = 1$, while if $|\lambda_i| > 2$ then the two values of μ_i are real, and given by $\mu_i = \frac{1}{2}(\lambda_i \pm \sqrt{\lambda_i^2 - 4})$. The larger of these has the plus sign; since this gives a monotonic increasing function of λ_i for $\lambda_i > 2$, it follows that the largest positive eigenvalue is $\mu = \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4})$, with multiplicity one, and all other eigenvalues of $\left(\begin{smallmatrix} E & -I \\ I & 0 \end{smallmatrix}\right)$ have strictly smaller absolute value. All the entries in the eigenvector $\left(\begin{smallmatrix} \mu v \\ v \end{smallmatrix}\right)$ are strictly positive. The corresponding row eigenvector is $(-\mu v^{\mathrm{T}}, v^{\mathrm{T}})$.

Proposition 3.5. *Suppose that $\lambda > 2$ and $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is any non-zero column vector such that for all $n \geqslant 0$, $\left(\begin{smallmatrix} E & -I \\ I & 0 \end{smallmatrix}\right)^n \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ has non-negative integer entries. If (δ, δ') is any row vector of length $2r$ with strictly positive integer entries then the power series*

$$\sum_{n=0}^{\infty} t^n (\delta, \delta') \begin{pmatrix} E & -I \\ I & 0 \end{pmatrix}^n \begin{pmatrix} x \\ y \end{pmatrix}$$

is a rational function of t with radius of convergence $R = \frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4})$. In particular, the growth rate of the coefficients of this power series is exponential.

Proof. Let v be a positive eigenvector of E with eigenvalue λ, as above. Write

$$\begin{pmatrix} x \\ y \end{pmatrix} = a \begin{pmatrix} \mu v \\ v \end{pmatrix} + b \begin{pmatrix} v \\ \mu v \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix}$$

where $v^{\mathrm{T}} x' = v^{\mathrm{T}} y' = 0$. This is possible since μ and μ^{-1} are simple roots of the characteristic equation of $\left(\begin{smallmatrix} E & -I \\ I & 0 \end{smallmatrix}\right)$. We claim that $a \neq 0$. For if $a = 0$ then

$$\begin{pmatrix} E & -I \\ I & 0 \end{pmatrix}^n \begin{pmatrix} x \\ y \end{pmatrix} = \mu^{-n} b \begin{pmatrix} v \\ \mu v \end{pmatrix} + \begin{pmatrix} E & -I \\ I & 0 \end{pmatrix}^n \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

The first term on the right-hand side tends to zero as $n \to \infty$. So the entries in the second term get closer and closer to non-negative integers. For any particular value of n, not all these integers can be zero. This contradicts the fact that

$$(v^{\mathrm{T}}, v^{\mathrm{T}}) \begin{pmatrix} E & -I \\ I & 0 \end{pmatrix}^n \begin{pmatrix} x' \\ y' \end{pmatrix} = 0,$$

since v has positive entries.

We have

$$\sum_{n=0}^{\infty} t^n (\delta, \delta') \begin{pmatrix} E & -I \\ I & 0 \end{pmatrix}^n \begin{pmatrix} x \\ y \end{pmatrix} = a \sum_{n=0}^{\infty} t^n (\delta, \delta') \begin{pmatrix} E & -I \\ I & 0 \end{pmatrix}^n \begin{pmatrix} \mu v \\ v \end{pmatrix}$$
$$+ \sum_{n=0}^{\infty} t^n (\delta, \delta') \begin{pmatrix} E & -I \\ I & 0 \end{pmatrix}^n \begin{pmatrix} bv + x' \\ b\mu v + y' \end{pmatrix}.$$

The first sum on the right-hand side is equal to

$$a\sum_{n=0}^{\infty} t^n(\delta,\delta')\begin{pmatrix}\mu^{n+1}v\\ \mu^n v\end{pmatrix} = a(\mu\delta+\delta')v\sum_{n=0}^{\infty}\mu^n t^n = \frac{a(\mu\delta+\delta')v}{1-\mu t}.$$

This is a rational function of t with radius of convergence $\mu^{-1} = \frac{1}{2}(\lambda - \sqrt{\lambda^2-4})$. The remaining sum breaks up into terms corresponding to smaller eigenvalues, which therefore give rational functions of the same form, but with larger radii of convergence. □

We can now prove the case of Theorem 1.1 for $\lambda > 2$. If M is a non-projective indecomposable Λ-module, then at most a finite number of the modules $\Omega^n M$ can be simple. So by replacing M by $\Omega^n M$ for n large enough, we may suppose that $\Omega^n M$ is not simple for $n \geqslant 0$. Let δ be the row vector whose ith entry is $\dim_k S_i$. Then using Eq. (2.1), we have

$$\dim_k \Omega^n M = (\delta,\delta)d\big(\Omega^n M\big) = (\delta,\delta)\begin{pmatrix}E & -I\\ I & 0\end{pmatrix}^n d(M).$$

So we can apply Proposition 3.5 to the Poincaré series

$$\sum_{n=0}^{\infty} t^n \dim_k \Omega^n M = \sum_{n=0}^{\infty} t^n(\delta,\delta)\begin{pmatrix}E & -I\\ I & 0\end{pmatrix}^n d(M)$$

to deduce the case $\lambda > 2$ of Theorem 1.1.

4. Eigenvalues at most two

Now suppose that $\lambda \leqslant 2$. We need the following theorem.

Theorem 4.1. *The only connected graphs with no loops, having maximum eigenvalue equal to* 2 *are* $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ *and* $\tilde{E}_8$. *The only connected graphs with no loops, having maximum eigenvalue strictly less than* 2 *are* A_n, D_n, E_6, E_7 *and* E_8.

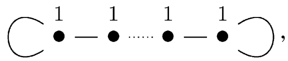
Proof. See J.H. Smith [13], or Theorem 5.1 of Lemmens and Seidel [7]. □

We need the following generalisation of Theorem 4.1.

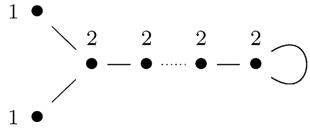
Theorem 4.2. *The only connected graphs having maximum eigenvalue equal to* 2 *are* $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{Z}_n$ *and* $\widetilde{DZ}_n$. *The only connected graphs having maximum eigenvalue strictly less than* 2 *are* A_n, D_n, E_6, E_7, E_8 *and* Z_n.

Proof. The proof is exactly the same as the proof of Theorem 4.1. The main steps are to find positive eigenvectors for $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{Z}_n$ or $\widetilde{DZ}_n$ with eigenvalue equal to 2, and to show that if a graph does not contain a subgraph isomorphic to one of these, then it is isomorphic to a proper subgraph of one of them. In the cases of $\tilde{Z}_n$ and $\widetilde{DZ}_n$, the eigenvectors are as follows:

$\tilde{Z}_n$:



$\widetilde{DZ}_n$:



So for example the adjacency matrix for $\tilde{Z}_3$ is $\begin{pmatrix} 1&1&0&0\\ 1&0&1&0\\ 0&1&0&1\\ 0&0&1&1 \end{pmatrix}$ and the eigenvector is $\begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$; the adjacency for $\widetilde{DZ}_3$ is $\begin{pmatrix} 0&0&1&0\\ 0&0&1&0\\ 1&1&0&1\\ 0&0&1&1 \end{pmatrix}$ and the eigenvector is $\begin{pmatrix} 1\\1\\2\\2 \end{pmatrix}$.

Finally, every proper subgraph of one of these graphs is of one of the Dynkin types A_n, D_n, E_6, E_7, E_8, or the graph Z_n. $\square$

Now we begin the proof of Theorem 1.1 in the cases where $\lambda \leqslant 2$. Theorem 4.2 shows that the matrix E has the type stated. All the eigenvalues μ of $\left(\begin{smallmatrix} E & -I \\ I & 0 \end{smallmatrix}\right)$ satisfy $|\mu| = 1$, by Lemma 3.4. Moreover, it follows from the discussion preceding Proposition 3.5 that the algebraic and geometric eigenspaces coincide, with one possible exception. The exception is that if $\lambda = 2$ then $\mu = 1$ occurs with algebraic multiplicity two and geometric multiplicity one. Thus if $\lambda < 2$ then the dimensions of the modules in the minimal projective resolution of each finitely generated Λ-module are bounded.

Now suppose that $\lambda = 2$, and that v is a positive eigenvector of E with eigenvalue 2. Suppose that M is an indecomposable module with an unbounded resolution. Choose m large enough so that for $n \geqslant m$, $\Omega^n M$ is not simple. Then for all $n \geqslant m$, $d(\Omega^n M)$ is an algebraic eigenvector of $\left(\begin{smallmatrix} E & -I \\ I & 0 \end{smallmatrix}\right)$ but not a geometric eigenvector. Thus $d(\Omega^m M)$ is of the form $a\left(\begin{smallmatrix} v \\ v \end{smallmatrix}\right) + b\left(\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right)$, where a and b are real numbers and $b \neq 0$. Then for $n \geqslant m$, Eq. (2.1) gives

$$d\big(\Omega^n M\big) = \big(a + (n-m)b\big)\begin{pmatrix} v \\ v \end{pmatrix} + b\begin{pmatrix} v \\ 0 \end{pmatrix}. \tag{4.3}$$

Since the entries in $d(\Omega^n M)$ have to be non-negative integers for all $n \geqslant m$, it follows that $b > 0$. Furthermore, Eq. (4.3) cannot hold for all $n \in \mathbb{Z}$, again by non-negativity. Using Eq. (2.1) again, the only possibility then is that for some $n \in \mathbb{Z}$, the module $\Omega^n M$ is simple.

Finally, we prove Corollary 1.2. By a theorem of Friedlander and Suslin [5], if Γ is a finite dimensional cocommutative Hopf algebra over a field k then the cohomology ring $H^*(\Gamma, k)$ is a finitely generated graded commutative k-algebra. Furthermore, if M is a finitely generated Γ-module then for each simple module S_i, $\mathrm{Ext}^*_\Gamma(S_i, M)$ is a finitely generated $H^*(\Gamma, k)$-module. It follows that the minimal projective resolution of M has polynomial growth. If Λ is a block of Γ with $J^3(\Lambda) = 0$ then Λ is also a symmetric algebra, and so it satisfies the hypotheses of Theorem 1.1. It follows that only cases (ii) and (iii) of the theorem are possible.

Acknowledgment

I would like to thank Avramov for pointing out to me that there have also been results known for a long time in the case of a commutative local Gorenstein ring with radical cubed equal

to zero. Sjödin [12] shows that the Poincaré series of every module is rational with denominator $1 - \lambda t + t^2$. Furthermore, the dichotomy between polynomial and exponential growth in this case was observed by Lescot [8], as well as the fact that in the polynomial case the degree is either zero or one. In a more recent paper of Avramov, Iyengar and Şega [2], it is observed that every indecomposable module is Koszul with the exception of the negative syzygies of the residue field.

References

[1] M. Auslander, I. Reiten, Sverre O. Smalø, Representation Theory of Artin Algebras, Cambridge Stud. Adv. Math., vol. 36, Cambridge University Press, 1995, reprinted in paperback, 1997.
[2] L.L. Avramov, S.B. Iyengar, L.M. Şega, Free resolutions over short local rings, arXiv:0707.4451.
[3] D.J. Benson, An algebraic model for chains on $\Omega(BG_p^\wedge)$, Trans. Amer. Math. Soc., in press.
[4] V. Dlab, C.M. Ringel, Indecomposable representations of graphs and algebras, Mem. Amer. Math. Soc. 173 (1976).
[5] E.M. Friedlander, A. Suslin, Cohomology of finite group schemes over a field, Invent. Math. 127 (1997) 209–270.
[6] G. Frobenius, Über Matrizen aus nicht negativen Elementen, Sitzungsber. Königl. Preuss. Akad. Wiss. Berlin (1912) 456–477.
[7] P.W. Lemmens, J.J. Seidel, Equiangular lines, J. Algebra 24 (1973) 494–512.
[8] J. Lescot, Asymptotic properties of Betti numbers of modules over certain rings, J. Pure Appl. Algebra 38 (1985) 287–298.
[9] T. Okuyama, On blocks of finite groups with radical cube zero, Osaka J. Math. 23 (1986) 461–465.
[10] O. Perron, Zur Theorie der Matrizen, Math. Ann. 64 (1907) 248–263.
[11] I. Reiten, Stable equivalence for some categories with radical square zero, Trans. Amer. Math. Soc. 212 (1975) 333–345.
[12] G. Sjödin, Poincaré series of modules over Gorenstein rings with $\mathfrak{m}^3 = 0$, Dept. of Maths, Stockholm Univ. Preprint No. 2, 1979.
[13] J.H. Smith, Some properties of the spectrum of a graph, in: R. Guy (Ed.), Combinatorial Structures and Their Applications, Gordon and Breach, New York, 1970, pp. 403–406.